Dynamics of dislocation densities in a bounded channel. Part II: existence of weak solutions to a singular Hamilton-Jacobi/parabolic strongly coupled system

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May 25, 2008

Abstract

We study a strongly coupled system of a parabolic equation and a singular Hamilton-Jacobi equation in one space dimension. This system describes the dynamics of dislocation densities in a material submitted to an exterior applied stress. Our system is a natural extension of that studied in [15] where the applied stress was set to be zero. The equations are written on a bounded interval and require special attention to the boundary layer. For this system, we prove a result of existence of a solution. The method of the proof consists in considering first a parabolic regularization of the full system, and then passing to the limit. For this regularized system, a result of global existence and uniqueness of a solution has been given in [16]. We show some uniform bounds on this solution which uses in particular an entropy estimate for the densities.

AMS Classification: 70H20, 49L25, 54C70, 46E30.
Key words: Hamilton-Jacobi equations, viscosity solutions, entropy, Orlicz spaces.

1 Introduction

1.1 Physical motivation and setting of the problem

In [12], Groma, Czikor and Zaiser have proposed a model describing the dynamics of dislocation densities. Dislocations are defects in crystals that move when a stress field is applied on the material. These defects are one of the main explanations of the elastoviscoplasticity behavior of metals (see [8] and [9] for various models relating dislocations and elastoviscoplastic preperties of metals). This model has been introduced to describe the possible accumulation of dislocations on the boundary layer of a bounded channel. More precisely, let us call θ^+ and θ^-, the densities of the positive and negative dislocations respectively. In fact, dislocations are distinguished by the sign of their Burgers vector $\vec{b}$

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(see [13] for a description of the Burgers vector). The non-negative densities $\theta^+(x,t)$ and $\theta^-(x,t)$ are governed by the following system:

$$\begin{cases} \theta_t^+ = \left[\left(\dfrac{\theta_x^+ - \theta_x^-}{\theta^+ + \theta^-} - \tau\right)\theta^+\right]_x & \text{in} \quad I \times (0,T), \\ \theta_t^- = \left[-\left(\dfrac{\theta_x^+ - \theta_x^-}{\theta^+ + \theta^-} - \tau\right)\theta^-\right]_x & \text{in} \quad I \times (0,T), \end{cases} \tag{1.1}$$

where τ is the stress field, $T > 0$, and $I := (0,1) \subset \mathbb{R}$. The channel is bounded by walls that are impenetrable by dislocations (i.e., the plastic deformation in the walls is zero). In this case the boundary conditions are represented by the zero flux condition, i.e.

$$\frac{\theta_x^+ - \theta_x^-}{\theta^+ + \theta^-} - \tau = 0, \quad \text{at} \quad x = 0 \ \text{ and } x = 1. \tag{1.2}$$

The original model in [12] is written in two space dimensions (x,y). Here, system (1.1) corresponds to a situation where the problem is assumed invariant by translation in the y direction. In that case τ appears to be the applied stress field and will be assumed to be a constant. However, the term $\frac{\theta_x^+ - \theta_x^-}{\theta^+ + \theta^-}$ is called the back stress and can be interpreted as the contribution to the stress of the short-range interactions between dislocations. This term was, for instance, neglected in the Groma-Balogh model [11]. Moreover, for the model described in [11], we refer the reader to [5, 6] for a one-dimensional mathematical and numerical study, and to [4] for a two-dimensional existence result. The special case $\tau = 0$ for system (1.1) has been studied in [15] where a result of existence and uniqueness has been proved. In the present paper we study the case where $\tau \neq 0$.

1.2 Setting of the problem

We consider an integrated form of (1.1) and we let

$$\rho_x^\pm = \theta^\pm, \quad \rho = \rho^+ - \rho^- \quad \text{and} \quad \kappa = \rho^+ + \rho^-,$$

to obtain (at least formally), for special values of the constants of integration, the following system in terms of ρ and κ:

$$\begin{cases} \kappa_t \kappa_x = \rho_t \rho_x & \text{on} \quad I \times (0,T) \\ \rho_t = \rho_{xx} - \tau \kappa_x & \text{on} \quad I \times (0,T), \end{cases} \tag{1.3}$$

with the initial conditions:

$$\kappa(x,0) = \kappa^0(x) \quad \text{and} \quad \rho(x,0) = \rho^0(x). \tag{1.4}$$

To formulate heuristically the boundary conditions at the walls located at $x = 0$ and $x = 1$, we suppose that $\kappa_x \neq 0$ at $x = 0$ and $x = 1$. We note that the dislocation fluxes at the walls must be zero, which require (see 1.2) that :

$$\overbrace{(\theta_x^+ - \theta_x^-) - \tau(\theta^+ + \theta^-)}^{\Phi} = 0, \quad \text{at} \quad x = 0 \ \text{ and } x = 1. \tag{1.5}$$

Rewriting system (1.3) in terms of ρ, κ and Φ, we get

$$\begin{cases} \kappa_t = (\rho_x/\kappa_x)\Phi, \\ \rho_t = \Phi. \end{cases} \tag{1.6}$$

From (1.5) and (1.6), we deduce that

$$\rho_t(0,.) = \rho_t(1,.) = 0. \tag{1.7}$$

Also, from (1.5) and (1.6), and if $\kappa_x \neq 0$ at $x = 0$ and $x = 1$, we deduce that

$$\kappa_t(0,.) = \kappa_t(1,.) = 0. \tag{1.8}$$

Using (1.7) and (1.8), we can formally reformulate the boundary conditions as follows:

$$\begin{cases} \kappa(0,.) = \kappa^0(0) \quad \text{and} \quad \kappa(1,.) = \kappa^0(1), \\ \rho(0,.) = \rho(1,.) = 0, \end{cases} \tag{1.9}$$

where we have taken the zero normalization for ρ on the boundary of the interval.

The positivity of θ^+ and θ^- reduces in terms of ρ and κ to the following condition:

$$\kappa_x \geq |\rho_x|, \tag{1.10}$$

and hence a natural assumption to be considered concerning the initial conditions ρ^0 and κ^0 is to satisfy

$$\kappa_x^0 \geq |\rho_x^0| \quad \text{on} \quad I. \tag{1.11}$$

Problem (1.3), (1.4) and (1.9), in the case $\tau = 0$, has been studied in [15] where a result of existence and uniqueness is given using the viscosity/entropy solution framework. Let us just mention that in this situation, system (1.3) becomes decoupled and easier to be handled.

1.3 Statement of the main result

In this paper, we assume that τ is a real constant,

$$\tau \neq 0$$

and we examine the existence of solutions of (1.3), (1.4) and (1.9). To be more precise, our main result is:

Theorem 1.1 ***(Existence of a solution)***
Let $\rho^0, \kappa^0 \in C^\infty(\bar{I})$ satisfying (1.11), (1.9) and the additional conditions:

$$D_x^s \rho^0 = D_x^s \kappa^0 = 0, \quad s = 1, 2, \quad x = 0, 1. \tag{1.12}$$

Then for every $T > 0$, there exists

$$(\rho, \kappa) \in (C(\bar{I} \times [0,T]))^2 \quad \textit{and} \quad \rho \in C^1(I \times (0,T)),$$

solution of (1.3), (1.4) and (1.9). Moreover, this solution satisfies (1.10) in the distributional sense, i.e.

$$\kappa_x \geq |\rho_x| \quad in \quad \mathcal{D}'(I \times (0,T)). \tag{1.13}$$

However, the solution has to be interpreted in the following sense:

1. *κ is a viscosity solution of $\kappa_t \kappa_x = \rho_t \rho_x$ in $I_T := I \times (0,T)$,*
2. *ρ is a distributional solution of $\rho_t = \rho_{xx} - \tau \kappa_x$ in I_T,*
3. *the initial and boundary conditions are satisfied pointwisely.*

Remark 1.2 ***(Compatibility of the regularized solution)***
The method of the proof of Theorem 1.1 consists in considering a parabolic regularization of (1.3), and then passing to the limit. This method is called the "vanishing viscosity" method. We use a result of global existence and uniqueness of the regularized system from [16], which requires some compatibility conditions on the initial data of the problem. The additional boundary conditions (1.12) was taken for achieving the compatibility at the regularized level.

Remark 1.3 *The C^∞ regularity of ρ^0 and κ^0, together with (1.12) seems to be essentially technical.*

Vanishing viscosity method is common in order to approach viscosity solutions for a Hamilton-Jacobi equation. It consists to add $\varepsilon\Delta$ to the Hamilton-Jacobi equation $H(x,u,Du) = 0$ and then obtain a more standard parabolic equation, after that we need to pass to the limit $\varepsilon \to 0$. The literature is very rich and one can cite for instance the Book of Barles [2] and the references therein, see also [20, 14].

In our case, we are interested in a singular Hamilton-Jacobi equation, strongly coupled with a parabolic equation. The singularity comes from the following formal formulation of the first equation of (1.3):

$$\kappa_t = \frac{\rho_t \rho_x}{\kappa_x},$$

that becomes a singular parabolic equation after adding the $\varepsilon\Delta$ term:

$$\kappa_t = \frac{\rho_t \rho_x}{\kappa_x} + \varepsilon \kappa_{xx}.$$

For a mathematical treatement of the above equation and various singular parabolic equations, see [16] and the references therein.

1.4 Organization of the paper

This paper is organized as follows: in section 2, we present the strategey of the proof. In section 3, we present the tools needed throughout this work. This includes some miscellaneous results for parabolic equations; a brief recall to the definition and the stability result of viscosity solutions; and a brief recall to Orlicz spaces. In section 4, we show how to choose the regularized solution. An entropy inequality used to determine some uniform bounds on the regularized solution is presented in section 5. Further uniform bounds and convergence arguments are done in section 6. Section 7 is devoted to the prove of our main result: Theorem 1.1. In section 8, some mumerical simulations related to our physical model are presented. Finally, section 9 is an appendix where we show the proofs of some standard results.

2 Strategy of the proof

The main difficulty we have to face is to work with the equation

$$\kappa_t \kappa_x = \rho_t \rho_x. \tag{2.1}$$

Since ρ solves itself a parabolic equation (see (1.3)), we expect enough regularity on ρ (indeed ρ is C^1), and then we need a framework where the equation on κ is stable under approximation. This property is naturally satisfied in the framework of viscosity solutions. Then, assuming $\kappa_x \geq 0$, we interpret κ as the viscosity solution of (2.1). Assuming (1.11), we will indeed show that

$$M := \kappa_x - |\rho_x| \geq 0.$$

This is formally true because M formally satisfies:

$$M_t = bM_x + cM,$$

with

$$b = \tau \operatorname{sgn}(\rho_x) - \frac{\rho_x \rho_{xx}}{\kappa_x^2}, \quad c = \frac{\rho_{xx}^2}{\kappa_x^2} - \frac{\rho_{xxx} \operatorname{sgn}(\rho_x)}{\kappa_x},$$

where for suitable boundary conditions, we can (again formally) see that

$$M \geq 0.$$

In order to justify the computations on M, we modify the system and we consider the following parabolic regularization for $\varepsilon > 0$ small enough:

$$\begin{cases} \kappa_t^\varepsilon = \varepsilon \kappa_{xx}^\varepsilon + \dfrac{\rho_x^\varepsilon \rho_{xx}^\varepsilon}{\kappa_x^\varepsilon} - \tau \rho_x^\varepsilon & \text{in} \quad I \times (0,T) \\ \rho_t^\varepsilon = (1+\varepsilon)\rho_{xx}^\varepsilon - \tau \kappa_x^\varepsilon & \text{in} \quad I \times (0,T), \end{cases} \tag{2.2}$$

which formally reduces to (1.3) for $\varepsilon = 0$, with initial conditions (1.4) and boundary conditions (1.9). In fact, system (2.2), (1.4) , (1.9), and under some conditions on the initial and boundary data, has a unique smooth global solution (see [16, Theorem **??**]) for $\alpha \in (0,1)$:

$$(\rho^\varepsilon, \kappa^\varepsilon) \in C^{3+\alpha, \frac{3+\alpha}{2}}(\bar{I} \times [0,\infty)) \cap C^\infty(I \times (0,\infty)).$$

This result will be clearly presented in the tools (see Theorem 3.1, Section 3). The next step is to find some uniform bounds (independent of ε) on this solution; this is done via:

(1) an entropy inequality shown to be valid for our special approximated model (2.2);

(2) a bound on $\kappa_t^\varepsilon - \varepsilon\kappa_{xx}^\varepsilon$ uniformly in ε.

In fact, (1) guarantees the global uniform-in-time control of the modulus of continuity in space of our approximated solution, while (2) guarantees the local uniform-in-space control of the modulus of continuity in time. The entropy inequality can be easily understood. For instance, for $\varepsilon = 0$ and $\tau = 0$, we can formally check that the entropy of the dislocation densities

$$\theta^\pm = \frac{\kappa_x \pm \rho_x}{2},$$

defined by:

$$S(t) = \int_I \sum_\pm \theta^\pm(.,t)\log(\theta^\pm(.,t)),$$

satisfies:

$$\frac{dS(t)}{dt} = -\int_I \frac{(\theta_x^+ - \theta_x^-)^2}{\theta^+ + \theta^-} \leq 0.$$

Therefore we get $S(t) \leq S(0)$ which controls the entropy uniformly in time. Finally, we need to pass to the limit $\varepsilon \to 0$ in the approximated solution after multiplying the first equation of (2.2) by κ_x^ε. Having enough control on the approximared solutions, we can find a solution of the limit equation using in particular the stability of viscosity solutions of Hamilton-Jacobi equations. However, the passage to the limit in the second equation of (2.2) is done in the distributional sense.

3 Tools: miscellaneous parabolic results, viscosity solution, and Orlicz spaces

3.1 Miscellaneous parabolic results

We first fix some notations. Denote

$$I_T := I \times (0,T), \quad \overline{I_T} := \bar{I} \times [0,T] \quad \text{and} \quad \partial^p I_T := I \cup (\partial I \times [0,T]).$$

Define the Sobolev space $W_p^{2,1}(I_T)$, $1 < p < \infty$ by:

$$W_p^{2,1}(I_T) := \left\{ u \in L^p(I_T);\ (u_t, u_x, u_{xx}) \in (L^p(I_T))^3 \right\}.$$

We start with a result of global existence and uniqueness of smooth solutions of the regularised system (2.2), with the initial and boundary conditions (1.4) and (1.9).

Theorem 3.1 ***(Global existence for the regularized system, [16, Theorem 1.1])*** *Let $0 < \alpha < 1$ and $0 < \varepsilon < 1$. Let $\rho^{0,\varepsilon}$, $\kappa^{0,\varepsilon}$ satisfying:*

$$\rho^{0,\varepsilon}, \kappa^{0,\varepsilon} \in C^\infty(\bar{I}), \quad \rho^{0,\varepsilon}(0) = \rho^{0,\varepsilon}(1) = \kappa^{0,\varepsilon}(0) = 0, \quad \kappa^{0,\varepsilon}(1) = 1, \tag{3.1}$$

$$\begin{cases} (1+\varepsilon)\rho_{xx}^{0,\varepsilon} = \tau\kappa_x^{0,\varepsilon} & on \quad \partial I \\ (1+\varepsilon)\kappa_{xx}^{0,\varepsilon} = \tau\rho_x^{0,\varepsilon} & on \quad \partial I, \end{cases} \tag{3.2}$$

and

$$\min_{x\in I}\left(\kappa_x^{0,\varepsilon}(x)-|\rho_x^{0,\varepsilon}(x)|\right)>0. \tag{3.3}$$

Then there exists a unique global solution

$$(\rho^\varepsilon,\kappa^\varepsilon)\in C^{3+\alpha,\frac{3+\alpha}{2}}(\bar I\times[0,\infty))\cap C^\infty(I\times(0,\infty)), \tag{3.4}$$

of the system (2.2), (1.4) and (1.9). Moreover, this solution satisfies :

$$\kappa_x^\varepsilon>|\rho_x^\varepsilon| \quad on \quad \bar I\times[0,\infty). \tag{3.5}$$

Remark 3.2 *Conditions (3.2) are natural here. Indeed, the regularity (3.4) of the solution of equation (2.2) with boundary conditions (1.4) and (1.9) imply in particular condition (3.2).*

Remark 3.3 ***(Uniform L^∞ bound on ρ^ε and κ^ε)***
We remark, from the boundary conditions (1.9) and from the inequality (3.5), that:

$$\|\rho^\varepsilon\|_{L^\infty(\bar I\times[0,\infty))}\le 1 \quad and \quad \|\kappa^\varepsilon\|_{L^\infty(\bar I\times[0,\infty))}\le 1. \tag{3.6}$$

We now present two technical lemmas that will be used in the proof of Theorem 1.1. The proofs of these lemmas will be given in the Appendix.

Lemma 3.4 ***(Control of the modulus of continuity in time uniformly in ε)***
Let $p>3$, and

$$u^\varepsilon\in W_p^{2,1}(I_T). \tag{3.7}$$

Suppose furthermore that the sequences

$$(u^\varepsilon)_\varepsilon \quad and \quad (f^\varepsilon)_\varepsilon=(u_t^\varepsilon-\varepsilon u_{xx}^\varepsilon)_\varepsilon, \tag{3.8}$$

are locally bounded in I_T uniformly for $\varepsilon\in(0,1)$. Then for every $V\subset\subset I_T$, there exist two constants $c>0$, $\varepsilon_0>0$ depending on V, and $0<\beta<1$ such that for all $0<\varepsilon<\varepsilon_0$:

$$\frac{|u^\varepsilon(x,t+h)-u^\varepsilon(x,t)|}{h^\beta}\le c, \quad \forall (x,t),(x,t+h)\in V. \tag{3.9}$$

Lemma 3.5 ***(An interior estimate for the heat equation)***
let $a\in C^\infty(I_T)\cap L^1(I_T)$ satisfying:

$$a_t=a_{xx} \quad on \quad I_T, \tag{3.10}$$

then for any $V\subset\subset I_T$, an open set, we have:

$$\|a\|_{p,V}\le c\|a\|_{1,I_T}, \quad \forall\, 1<p<\infty, \tag{3.11}$$

with $c=c(p,V)>0$ is a positive constant.

3.2 Viscosity solution: definition and stability result

Let $\Omega \subset \mathbb{R}^n$ be an open domain, and consider the following Hamilton-Jacobi equation:

$$F(x, u(x), Du(x), D^2u(x)) = 0, \quad \forall x \in \Omega, \tag{3.12}$$

where $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times M^{n\times n}_{sym} \mapsto \mathbb{R}$ is a continuous mapping.

Definition 3.6 ***(Viscosity solution of Hamilton-Jacobi equations)***
A continuous function $u : \Omega \mapsto \mathbb{R}$ is a viscosity sub-solution of (3.12) if for any $\phi \in C^2(\Omega; \mathbb{R})$ and any local maximum $x_0 \in \Omega$ of $u - \phi$, one has

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \leq 0.$$

Similarly, u is a viscosity super-solution of (3.12), if at any local minimum point $x_0 \in \Omega$ of $u - \phi$, one has

$$F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0.$$

Finally, if u is both a viscosity sub-solution and a viscosity super-solution, then u is called a viscosity solution.

To get a "non-empty" and useful definition, it is usually assumed that F is elliptic (see [2]). This notion of ellipticity will be indirectly used in Section 7. In fact, this definition is used for interpreting solutions of the first equation of (1.3) in the viscosity sense. This will be shown in Section 5. To be more precise, in the case where $\Omega = I_T$, we say that u is a viscosity solution of the Dirichlet problem (3.12) with $u = \zeta \in C(\partial^p I_T)$ if:

(1) $u \in C(\overline{I_T})$,

(2) u is a viscosity solution of (3.12) in I_T,

(3) $u = \zeta$ on $\partial^p I_T$.

For a better understanding of the viscosity interpretation of boundary conditions of Hamilton-Jacobi equations, we refer the reader to [2, Section 4.2]. We now state the stability result for viscosity solutions of Hamilton-Jacobi equations. An important result concerning viscosity solutions is presented by the following theorem:

Theorem 3.7 ***(Stability of viscosity solutions, [2, Lemma 2.3])***
Suppose that, for $\varepsilon > 0$, $u^\varepsilon \in C(\Omega)$ is a viscosity sub-solution (resp. super-solution) of the equation

$$H^\varepsilon(x, u^\varepsilon, Du^\varepsilon, D^2u^\varepsilon) = 0 \quad in \quad \Omega, \tag{3.13}$$

where $(H^\varepsilon)_\varepsilon$ is a sequence of continuous functions. If $u^\varepsilon \to u$ locally uniformly in Ω and if $H^\varepsilon \to H$ locally uniformly in $\Omega \times \mathbb{R} \times \mathbb{R}^n \times M^{n\times n}_{sym}$, then u is a viscosity sub-solution (resp. super-solution) of the equation:

$$H(x, u, Du, D^2u) = 0 \quad in \quad \Omega. \tag{3.14}$$

3.3 Orlicz spaces: definition and properties

We recall the definition of an Orlicz space and some of its properties (for details see [1]). A real valued function $\Psi : [0,\infty) \to \mathbb{R}$ is called a Young function if

$$\Psi(t) = \int_0^t \psi(s)ds,$$

where $\psi : [0,\infty) \to [0,\infty)$ satisfying:

- $\psi(0) = 0$, $\psi > 0$ on $(0,\infty)$, $\psi(t) \to \infty$ as $t \to \infty$;
- ψ is non-decreasing and right continuous at any point $s \geq 0$.

Let Ψ be a Young function. The Orlicz class $K_\Psi(I)$ is the set of equivalence classes of real-valued measurable functions u on I satisfying

$$\int_I \Psi(|u(x)|)dx < +\infty.$$

Definition 3.8 ***(Orlicz spaces)***
The Orlicz space $L_\Psi(I)$ is the linear span of $K_\Psi(I)$ supplemented with the Luxemburg norm

$$\|u\|_{L_\Psi(I)} = \inf\left\{k > 0;\ \int_I \Psi\left(\frac{|u(x)|}{k}\right) \leq 1\right\}, \tag{3.15}$$

and with this norm, the Orlicz space is a Banach space.

The function

$$\Phi(t) = \int_0^t \phi(s)ds, \quad \phi(s) = \sup_{\psi(t)\leq s} t,$$

is called the complementary Young function of Ψ. An example of such pair of complementary Young functions is the following:

$$\Psi(s) = (1+s)\log(1+s) - s \quad \text{and} \quad \Phi(s) = e^s - s - 1. \tag{3.16}$$

We now state a lemma giving two useful properties of Orlicz spaces that will be used in the proof of Lemma 5.4.

Lemma 3.9 ***(Norm control and Hölder inequality, [17])***
If $u \in L_\Psi(I)$ for some Young function Ψ, then we have:

$$\|u\|_{L_\Psi(I)} \leq 1 + \int_I \Psi(|u(x)|)dx. \tag{3.17}$$

Moreover, if $v \in L_\Phi(I)$, Φ being the complementary Young function of Ψ, then we have the following Hölder inequality:

$$\left|\int_I uvdx\right| \leq 2\|u\|_{L_\Psi(I)}\|v\|_{L_\Phi(I)}. \tag{3.18}$$

4 The regularized problem

As we have already mentioned, we will use a parabolic regularization of (1.3), and a result of global existence of this regularized system from [16] (see Theorem 3.1). In order to use this result, we need to give a special attention to the conditions on the initial data of the approximated system $\rho^{0,\varepsilon}$ and $\kappa^{0,\varepsilon}$ (see (3.1), (3.2) and (3.3)). This section aims to show how to choose the suitable initial data $\rho^{0,\varepsilon}$ and $\kappa^{0,\varepsilon}$ in order to benefit Theorem 3.1.

Let ρ^0 and κ^0 be the functions given in Theorem 1.1. Set

$$\rho^{0,\varepsilon} = \frac{\rho^0 + \varepsilon\tau\phi}{(1+\varepsilon)^2}, \tag{4.1}$$

and

$$\kappa^{0,\varepsilon} = \frac{\kappa^0 + \varepsilon x}{1+\varepsilon}, \tag{4.2}$$

with the function ϕ defined by:

$$\phi(x) = \frac{1}{\tau^2}[1 - \cos\tau(x^2 - x)]. \tag{4.3}$$

The function ϕ enjoys some properties that are shown in the following lemma.

Lemma 4.1 ***(Properties of ϕ)***
The function ϕ given by (4.3) satisfies the following properties:

(P1) $\phi, \phi'|_{\partial I} = 0$;

(P2) $\phi''|_{\partial I} = 1$;

(P3) $|\phi'(x)| < 1/\tau \quad$ for $\quad x \in \bar{I}$.

Proof. (P1) and (P2) directly follows by simple computations. For (P3), we calculate on $\bar{I}$:

$$\begin{aligned} |\phi'(x)| &= (1/\tau)|2x-1||\sin\tau(x^2-x)| \\ &\leq 1/\tau. \end{aligned}$$

In order to obtain the strict inequality, we remark that

$$|2x-1||\sin\tau(x^2-x)| \neq 1 \quad \text{on} \quad \bar{I},$$

hence $|\phi'(x)| < 1/\tau$. □

Form the above lemma, and from the construction of $\rho^{0,\varepsilon}$ and $\kappa^{0,\varepsilon}$ (see (4.1) and (4.2)) together with the properties enjoyed by ρ^0 and κ^0 (see (1.9) and (1.12)), we write down some properties of $\rho^{0,\varepsilon}$ and $\kappa^{0,\varepsilon}$.

Lemma 4.2 ***(Properties of*** $\rho^{0,\varepsilon}$ ***and*** $\kappa^{0,\varepsilon}$***)***
The functions $\rho^{0,\varepsilon}$ *and* $\kappa^{0,\varepsilon}$ *given respectively by (4.1) and (4.2) , satisfy the following properties:*

(P4) $\rho^{0,\varepsilon}(0) = \rho^{0,\varepsilon}(1) = \kappa^{0,\varepsilon}(0) = 0 \quad$ *and* $\quad \kappa^{0,\varepsilon}(1) = 1;$

(P5) $(1+\varepsilon)\kappa^{0,\varepsilon}_{xx}\big|_{\partial I} = \tau\rho^{0,\varepsilon}_{x}\big|_{\partial I} \quad$ *and* $\quad (1+\varepsilon)\rho^{0,\varepsilon}_{xx}\big|_{\partial I} = \tau\kappa^{0,\varepsilon}_{x}\big|_{\partial I};$

(P6) $\kappa^{0,\varepsilon}_{x} \geq |\rho^{0,\varepsilon}_{x}| + \dfrac{\varepsilon(1-\tau|\phi'|)}{1+\varepsilon} > |\rho^{0,\varepsilon}_{x}|.$

Proof. We only show (P5) and (P6). For (P5), we calculate:

$$\rho^{0,\varepsilon}_{x} = \frac{\rho^{0}_{x} + \varepsilon\tau\phi'}{(1+\varepsilon)^2}, \quad \rho^{0,\varepsilon}_{xx} = \frac{\rho^{0}_{xx} + \varepsilon\tau\phi''}{(1+\varepsilon)^2}, \tag{4.4}$$

and

$$\kappa^{0,\varepsilon}_{x} = \frac{\kappa^{0}_{x} + \varepsilon}{1+\varepsilon}, \quad \kappa^{0,\varepsilon}_{xx} = \frac{\kappa^{0}_{xx}}{1+\varepsilon}.$$

Therefore, on ∂I, we have:

$$(1+\varepsilon)\rho^{0,\varepsilon}_{xx} = \tau\left(\frac{\varepsilon}{1+\varepsilon}\right) = \tau\kappa^{0,\varepsilon}_{x},$$

and

$$(1+\varepsilon)\kappa^{0,\varepsilon}_{xx} = \tau\rho^{0,\varepsilon}_{x} = 0,$$

where we have used (P1) and (P2) from Lemma 4.1, and the properties (1.9), (1.12) of ρ^0 and κ^0 on ∂I. For (P6), we proceed as follows. We first use the inequality (1.11) between ρ^0_x and κ^0_x, to deduce that:

$$\kappa^{0,\varepsilon}_{x} = \frac{\kappa^{0}_{x} + \varepsilon}{1+\varepsilon} \geq \frac{|\rho^{0}_{x}| + \varepsilon}{1+\varepsilon},$$

and then from the left identity of (4.4), we deduce that:

$$\rho^{0}_{x} = (1+\varepsilon)^2\rho^{0,\varepsilon}_{x} - \varepsilon\tau\phi',$$

therefore

$$\kappa^{0,\varepsilon}_{x} \geq (1+\varepsilon)|\rho^{0,\varepsilon}_{x}| + \frac{\varepsilon(1-\tau|\phi'|)}{1+\varepsilon}.$$

The inequality (P6) then directly follows. □

Remark 4.3 ***(The regularized solution*** $(\rho^{\varepsilon}, \kappa^{\varepsilon})$***)***
Properties (P4)-(P5)-(P6) of Lemma 4.2 implies condition (3.1)-(3.2)-(3.3) of Theorem 3.1. In this case, call

$$(\rho^{\varepsilon}, \kappa^{\varepsilon}), \tag{4.5}$$

the solution of (2.2), (1.4) and (1.9), given in Theorem 3.1, with the initial conditions

$$\rho(x,0) = \rho^{0,\varepsilon} \quad \textit{and} \quad \kappa(x,0) = \kappa^{0,\varepsilon},$$

that are given by (4.1) and (4.2) respectively.

5 Entropy inequality

Proposition 5.1 ***(Entropy inequality)***
Let $(\rho^\varepsilon, \kappa^\varepsilon)$ be the regular solution given by (4.5). Define $\theta^{\pm,\varepsilon}$ by:

$$\theta^{\pm,\varepsilon} = \frac{\kappa_x^\varepsilon \pm \rho_x^\varepsilon}{2}, \tag{5.1}$$

then the quantity $S(t)$ given by:

$$S(t) = \int_I \sum_{\pm} \theta^{\pm,\varepsilon}(x,t) \log \theta^{\pm,\varepsilon}(x,t) dx, \tag{5.2}$$

satisfies for every $t \geq 0$:

$$S(t) \leq S(0) + \frac{\tau^2 t}{4}. \tag{5.3}$$

Proof. From (3.5), we know that

$$\kappa_x^\varepsilon > |\rho_x^\varepsilon|,$$

hence

$$\theta^{\pm,\varepsilon} > 0,$$

and the term $\log(\theta^{\pm,\varepsilon})$ is well defined. Also from the regularity (3.4) of the solution $(\rho^\varepsilon, \kappa^\varepsilon)$, we know that

$$\theta^{\pm,\varepsilon}(.,t) \in C(\bar{I}), \quad \forall t \geq 0,$$

hence the term $S(t)$ is well defined. We derive system (2.2) with respect to x, and we write it in terms of $\theta^{\pm,\varepsilon}$, we get:

$$\begin{cases} \theta_t^{+,\varepsilon} = \left[\left(\dfrac{(\theta^{+,\varepsilon} - \theta^{-,\varepsilon})_x}{\theta^{+,\varepsilon} + \theta^{-,\varepsilon}} - \tau\right)\theta^{+,\varepsilon} + \varepsilon\theta_x^{+,\varepsilon}\right]_x \\ \theta_t^{-,\varepsilon} = \left[-\left(\dfrac{(\theta^{+,\varepsilon} - \theta^{-,\varepsilon})_x}{\theta^{+,\varepsilon} + \theta^{-,\varepsilon}} - \tau\right)\theta^{-,\varepsilon} + \varepsilon\theta_x^{-,\varepsilon}\right]_x. \end{cases} \tag{5.4}$$

We first remark that:

$$\left(\frac{(\theta^{+,\varepsilon} - \theta^{-,\varepsilon})_x}{\theta^{+,\varepsilon} + \theta^{-,\varepsilon}} - \tau\right)\theta^{+,\varepsilon} + \varepsilon\theta_x^{+,\varepsilon} = \frac{\kappa_t + \rho_t}{2}$$

and

$$-\left(\frac{(\theta^{+,\varepsilon} - \theta^{-,\varepsilon})_x}{\theta^{+,\varepsilon} + \theta^{-,\varepsilon}} - \tau\right)\theta^{-,\varepsilon} + \varepsilon\theta_x^{-,\varepsilon} = \frac{\kappa_t - \rho_t}{2}.$$

Since κ_t^ε and ρ_t^ε are zeros on $\partial I \times [0,\infty)$, then

$$\left(\frac{(\theta^{+,\varepsilon} - \theta^{-,\varepsilon})_x}{\theta^{+,\varepsilon} + \theta^{-,\varepsilon}} - \tau\right)\theta^{+,\varepsilon} + \varepsilon\theta_x^{+,\varepsilon} = -\left(\frac{(\theta^{+,\varepsilon} - \theta^{-,\varepsilon})_x}{\theta^{+,\varepsilon} + \theta^{-,\varepsilon}} - \tau\right)\theta^{-,\varepsilon} + \varepsilon\theta_x^{-,\varepsilon} = 0 \text{ on } \partial I \times [0,\infty). \tag{5.5}$$

Using (5.5), we compute for $t \geq 0$:

$$\begin{aligned} S'(t) &= \sum_{\pm} \int_I \theta_t^{\pm,\varepsilon} \log(\theta^{\pm,\varepsilon}) + \theta_t^{\pm,\varepsilon}, \\ &= \sum_{\pm} \int_I \mp \left(\frac{(\theta^{+,\varepsilon} - \theta^{-,\varepsilon})_x}{\theta^{+,\varepsilon} + \theta^{-,\varepsilon}} - \tau \right) \theta_x^{\pm,\varepsilon} - \varepsilon \frac{\left(\theta_x^{\pm,\varepsilon}\right)^2}{\theta^{\pm,\varepsilon}}, \\ &= \int_I - \frac{\left(\theta_x^{+,\varepsilon} - \theta_x^{-,\varepsilon}\right)^2}{\theta^{+,\varepsilon} + \theta^{-,\varepsilon}} + \tau(\theta_x^{+,\varepsilon} - \theta_x^{-,\varepsilon}) - \varepsilon \left(\frac{\left(\theta_x^{+,\varepsilon}\right)^2}{\theta^{+,\varepsilon}} + \frac{\left(\theta_x^{-,\varepsilon}\right)^2}{\theta^{-,\varepsilon}} \right). \end{aligned}$$

By Young's Inequality, we have:

$$\left|\theta_x^{+,\varepsilon} - \theta_x^{-,\varepsilon}\right| \leq \frac{1}{\tau} \frac{\left(\theta_x^{+,\varepsilon} - \theta_x^{-,\varepsilon}\right)^2}{\theta^{+,\varepsilon} + \theta^{-,\varepsilon}} + \frac{\tau}{4}(\theta^{+,\varepsilon} + \theta^{-,\varepsilon}),$$

and hence

$$\begin{aligned} S'(t) &\leq \int_I \frac{\tau^2}{4}(\theta^{+,\varepsilon} + \theta^{-,\varepsilon}) - \varepsilon \left(\frac{\left(\theta_x^{+,\varepsilon}\right)^2}{\theta^{+,\varepsilon}} + \frac{\left(\theta_x^{-,\varepsilon}\right)^2}{\theta^{-,\varepsilon}} \right) \\ &\leq \frac{\tau^2}{4} \int_I (\theta^{+,\varepsilon} + \theta^{-,\varepsilon}). \end{aligned}$$

Moreover, we have from (1.9), that

$$\int_I (\theta^{+,\varepsilon}(.,t) + \theta^{-,\varepsilon}(.,t)) = \int_I \kappa_x(.,t) = \kappa(1,t) - \kappa(0,t) = 1,$$

and therefore

$$S'(t) \leq \frac{\tau^2}{4}.$$

Integrating the previous inequality from 0 to t, we get (5.3). □

An immediate corollary of Proposition 5.1 is the following:

Corollary 5.2 ***(Special control of κ_x^ε)***
For all $t \geq 0$, we have:

$$\int_I \kappa_x^\varepsilon(x,t) \log(\kappa_x^\varepsilon(x,t)) dx \leq S(0) + \frac{\tau^2 t}{4} + 1, \tag{5.6}$$

where S is given by (5.2).

The proof of Corollary 5.2 depends on the inequality shown by the next lemma.

Lemma 5.3 *For every $x, y > 0$, we have:*

$$(x+y)\log(x+y) \leq x\log(x) + y\log(y) + x\log(2) + y. \tag{5.7}$$

Proof. Fix $y > 0$. consider the function f defined by:

$$f(x) = (x + y)\log(x + y) - x\log(x) - y\log(y) - x\log(2) - y, \qquad x > 0. \tag{5.8}$$

We claim that $f(x) \leq 0$ for every $x > 0$. Indeed, we have $\lim_{x\to 0^+} f(x) = -y < 0$. We compute

$$f'(x) = \log(x + y) - \log(x) - \log(2), \tag{5.9}$$

and we remark that this is always a decreasing function with

$$\lim_{x\to 0^+} f'(x) = +\infty \quad \text{and} \quad \lim_{x\to +\infty} f'(x) = -\log(2),$$

hence the function $f(x)$ can only be positive if $f(x_0) > 0$ where x_0 satisfies

$$f'(x_0) = 0.$$

A simple computation shows that $x_0 = y$, then

$$\begin{aligned} f(y) &= 2y\log(2y) - 2y\log(y) - y\log(2) - y \\ &= 2y\log(2) + 2y\log(y) - 2y\log(y) - y\log(2) - y \\ &= y\log(2) - y < 0, \end{aligned}$$

and therefore $f(x) \leq 0$, $\forall x > 0$, which ends the proof. □

Proof of Corollary 5.2. From (5.1), it follows that

$$\kappa_x^\varepsilon = \theta^{+,\varepsilon} + \theta^{-,\varepsilon} > 0.$$

Then we have for $t \geq 0$:

$$\begin{aligned} \int_I \kappa_x^\varepsilon \log \kappa_x^\varepsilon &= \int_I (\theta^{+,\varepsilon} + \theta^{-,\varepsilon}) \log(\theta^{+,\varepsilon} + \theta^{-,\varepsilon}) \\ &\leq \int_I \theta^{+,\varepsilon} \log(\theta^{+,\varepsilon}) + \theta^{-,\varepsilon} \log(\theta^{-,\varepsilon}) + \theta^{+,\varepsilon} \log 2 + \theta^{-,\varepsilon} \\ &\leq \int_I \theta^{+,\varepsilon} \log(\theta^{+,\varepsilon}) + \theta^{-,\varepsilon} \log(\theta^{-,\varepsilon}) + \frac{1}{2}(\log 2 + 1) \\ &\leq S(t) + 1. \end{aligned}$$

Here we have used Lemma 5.3 with $x = \theta^{+,\varepsilon}$ and $y = \theta^{-,\varepsilon}$ for the second line, and we have used for the third line, the fact that

$$\int_I \theta^{\pm,\varepsilon} = \frac{1}{2} \int_I \kappa_x \pm \rho_x = \frac{1}{2}[\kappa(1, .) - \kappa(0, .)] = 1/2.$$

Using (5.3), the result follows. □

Lemma 5.4 ***(Control of the modulus of continuity in space)***
Let $u \in C^1(I)$, $u_x > 0$, satisfying

$$\int_I u_x \log(u_x) \leq c_1, \tag{5.10}$$

then we have for any $x,\ x+h \in I$:

$$|u(x+h)-u(x)| \leq \frac{c_2(1+c_1)}{|\log h|}, \tag{5.11}$$

where $c_2 >$ is a universal constant.

Proof. Let $x, x+h \in I$.

Step 1. ($u_x \in L_\Psi(x, x+h)$ with Ψ given in (3.16))

We compute

$$\begin{aligned}
\int_x^{x+h} \Psi(u_x) &= \int_x^{x+h} (1+u_x)\log(1+u_x) - u_x \\
&\leq \int_I (1+u_x)\log(1+u_x) - u_x \\
&\leq \int_I u_x \log(u_x) + \log 2 \\
&\leq c_1 + \log 2,
\end{aligned}$$

where we have used (5.7) in the third line, and (5.10) in the last line. Hence from (3.17), we get

$$\|u_x\|_{L_\Psi(x,x+h)} \leq c_1 + 1 + \log 2,$$

and hence $u_x \in L_\Psi(x, x+h)$.

Step 2. (Estimating the modulus of continuity)

It is easy to check that the function 1 lies in $L_\Phi(x, x+h)$, Φ is also given by (3.16). Therefore, by Hölder inequality (3.18), we obtain:

$$\begin{aligned}
|u(x+h)-u(x)| &= \left|\int_x^{x+h} u_x \cdot 1\right| \\
&\leq 2\|u_x\|_{L_\psi(x,x+h)}\|1\|_{L_\Phi(x,x+h)} \\
&\leq 2(c_1+1+\log 2)\|1\|_{L_\Phi(x,x+h)}.
\end{aligned} \tag{5.12}$$

We turn our attention now to the term $\|1\|_{L_\Phi(x,x+h)}$. We have

$$\begin{aligned}
\|1\|_{L_\Phi(x,x+h)} &= \inf\left\{k>0;\ \int_x^{x+h} \Phi\left(\frac{1}{k}\right) \leq 1\right\} \\
&= \inf\left\{k>0;\ \int_x^{x+h} (e^{1/k} - 1/k - 1) \leq 1\right\} \\
&= \inf\left\{k>0;\ h(e^{1/k} - 1/k - 1) \leq 1\right\} \\
&\leq -\frac{1}{\log(h)},
\end{aligned}$$

where we have used in the last line the fact that for $0 < h < 1$ and $k = -\frac{1}{\log(h)}$, the following inequality holds:

$$h(e^{1/k} - 1/k - 1) \leq 1.$$

Hence, (5.12) implies

$$|u(x+h) - u(x)| \leq 2(c_1 + 1 + \log 2)\frac{1}{|\log h|},$$

and then (5.11) follows. □

6 An interior estimate

In this section, we give an interior estimate for the term

$$A^\varepsilon = \rho_x^\varepsilon - \tau\kappa^\varepsilon. \tag{6.1}$$

that will be used in the passage to the limit as ε goes to zero in the regularized system. We start by deriving an equation satisfied by A^ε.

Lemma 6.1 *The quantity A^ε given by (6.1) satisfies for any $T > 0$:*

$$A_t^\varepsilon = (1+\varepsilon)A_{xx}^\varepsilon - \frac{\tau\rho_x^\varepsilon}{\kappa_x^\varepsilon}A_x^\varepsilon. \tag{6.2}$$

Proof. From (2.2), we calculate:

$$\begin{aligned}
A_t^\varepsilon &= \rho_{tx}^\varepsilon - \tau\kappa_t^\varepsilon \\
&= (1+\varepsilon)\rho_{xxx}^\varepsilon - \tau\kappa_{xx}^\varepsilon - \tau\left(\varepsilon\kappa_{xx}^\varepsilon + \frac{\rho_x^\varepsilon\rho_{xx}^\varepsilon}{\kappa_x^\varepsilon} - \tau\rho_x^\varepsilon\right) \\
&= (1+\varepsilon)(\rho_{xxx}^\varepsilon - \tau\kappa_{xx}^\varepsilon) - \frac{\tau\rho_x^\varepsilon}{\kappa_x^\varepsilon}(\rho_{xx}^\varepsilon - \tau\kappa_x^\varepsilon) \\
&= (1+\varepsilon)A_{xx}^\varepsilon - \frac{\tau\rho_x^\varepsilon}{\kappa_x^\varepsilon}A_x^\varepsilon,
\end{aligned}$$

hence (6.2) is satisfied. □

We now show an interior L^p estimate concerning the term A^ε. This estimate gives a control on the local L^p norm of A^ε by its global L^1 norm over I_T, and it will be used in the following section. More precisely, we have the following lemma.

Lemma 6.2 ***(Interior L^p estimate)***
Let $0 < \varepsilon < 1$ and $1 < p < \infty$. Then the quantity A^ε given by (6.1) satisfies:

$$\|A^\varepsilon\|_{p,V} \leq c\left(\|A^\varepsilon\|_{1,I_T} + 1\right), \tag{6.3}$$

where V is an open subset of I_T such that $V \subset\subset I_T$, and $c = c(p,V) > 0$ is a constant independent of ε.

Proof. Throughout the proof, the term $c = c(p, V) > 0$ is a positive constant independent of ε, and it may vary from line to line. A simple computation gives:

$$\begin{aligned}
-\tau \frac{\rho_x^\varepsilon}{\kappa_x^\varepsilon} A_x^\varepsilon &= -\tau \frac{\rho_x^\varepsilon}{\kappa_x^\varepsilon}(\rho_{xx}^\varepsilon - \tau\kappa_x^\varepsilon) \\
&= -\tau \frac{\rho_x^\varepsilon \rho_{xx}^\varepsilon}{\kappa_x^\varepsilon} + \tau^2 \rho_x^\varepsilon \\
&= -\tau(\kappa_t^\varepsilon - \varepsilon\kappa_{xx}^\varepsilon).
\end{aligned} \tag{6.4}$$

Define $\bar{\kappa}^\varepsilon$ as the unique solution of

$$\begin{cases} \bar{\kappa}_t^\varepsilon = (1+\varepsilon)\bar{\kappa}_{xx}^\varepsilon + \kappa^\varepsilon & \text{on} \quad I_T, \\ \bar{\kappa}^\varepsilon = 0 & \text{on} \quad \partial^p I_T, \end{cases} \tag{6.5}$$

where the existence and uniqueness of this equation is a direct consequence of the L^p theory for parabolic equations (see for instance [18, Theorem 9.1]) using in particular the fact that $\kappa^\varepsilon \in C^1(\overline{I_T})$. Moreover, from the regularity (3.4) of κ^ε, we can deduce that $\bar{\kappa}^\varepsilon \in C^\infty(I_T)$. Let $\bar{A}^\varepsilon$ be given by:

$$\bar{A}^\varepsilon = -\tau(\bar{\kappa}_t^\varepsilon - \varepsilon\bar{\kappa}_{xx}^\varepsilon), \tag{6.6}$$

and

$$a^\varepsilon = A^\varepsilon - \bar{A}^\varepsilon. \tag{6.7}$$

We calculate:

$$\begin{aligned}
\bar{A}_t^\varepsilon &= -\tau[\bar{\kappa}_{tt}^\varepsilon - \varepsilon\bar{\kappa}_{xxt}^\varepsilon] \\
&= -\tau[(1+\varepsilon)\bar{\kappa}_{xxt}^\varepsilon + \kappa_t^\varepsilon - \varepsilon((1+\varepsilon)\bar{\kappa}_{xxxx}^\varepsilon + \kappa_{xx}^\varepsilon)] \\
&= -\tau(1+\varepsilon)(\bar{\kappa}_{xxt}^\varepsilon - \varepsilon\bar{\kappa}_{xxxx}^\varepsilon) - \tau(\kappa_t^\varepsilon - \varepsilon\kappa_{xx}^\varepsilon) \\
&= (1+\varepsilon)\bar{A}_{xx} - \frac{\tau\rho_x^\varepsilon}{\kappa_x^\varepsilon} A_x^\varepsilon,
\end{aligned}$$

where for the first two line, we have used (6.5), and for the last line, we have used (6.4). In this case, we obtain:

$$\begin{aligned}
a_t^\varepsilon &= A_t^\varepsilon - \bar{A}_t^\varepsilon \\
&= (1+\varepsilon)A_{xx}^\varepsilon - \frac{\tau\rho_x^\varepsilon}{\kappa_x^\varepsilon} A_x^\varepsilon - (1+\varepsilon)\bar{A}_{xx} + \frac{\tau\rho_x^\varepsilon}{\kappa_x^\varepsilon} A_x^\varepsilon \\
&= (1+\varepsilon)(A_{xx}^\varepsilon - \bar{A}_{xx}^\varepsilon) \\
&= (1+\varepsilon)a_{xx}^\varepsilon,
\end{aligned}$$

where for the first line, we have used the equation (6.2). We apply Lemma 3.5 to the function a^ε, after doing parabolic rescaling of the form $\tilde{a}^\varepsilon(x,t) = a^\varepsilon\left(x, \frac{t}{1+\varepsilon}\right)$, we get:

$$\|a^\varepsilon\|_{p,V} \leq c(1+\varepsilon)^{1-\frac{1}{p}} \|a^\varepsilon\|_{1,I_T},$$

and since $0 < \varepsilon < 1$, we finally obtain

$$\|a^\varepsilon\|_{p,V} \leq c\|a^\varepsilon\|_{1,I_T}. \tag{6.8}$$

From the definition of a^ε (see (6.7) above), and the above inequality (6.8), we finally deduce that:

$$\|A^\varepsilon\|_{p,V} \leq c(\|A^\varepsilon\|_{1,I_T} + \|\bar{A}^\varepsilon\|_{p,I_T}). \tag{6.9}$$

In order to complet the proof, we need to control the term $\|\bar{A}^\varepsilon\|_{p,I_T}$ in (6.9). We use the equation (6.5) satisfied by $\bar{\kappa}^\varepsilon$ to obtain:

$$\begin{aligned}
\|\bar{A}^\varepsilon\|_{p,I_T} &= \tau\|\bar{\kappa}^\varepsilon_t - \varepsilon\bar{\kappa}^\varepsilon_{xx}\|_{p,I_T} \\
&= \tau\|\bar{\kappa}^\varepsilon_{xx} + \kappa^\varepsilon\|_{p,I_T} \\
&\leq c(\|\bar{\kappa}^\varepsilon_{xx}\|_{p,I_T} + \|\kappa^\varepsilon\|_{p,I_T}).
\end{aligned} \tag{6.10}$$

The L^p estimates for parabolic equations (see [16, Lemma **??**]) applied to (6.5) gives:

$$\|\bar{\kappa}^\varepsilon_{xx}\|_{p,I_T} \leq \frac{c}{1+\varepsilon}\|\kappa^\varepsilon\|_{p,I_T},$$

then (6.10), together with the fact that $0 \leq \kappa^\varepsilon \leq 1$ (see (1.9)), implies that:

$$\|\bar{A}^\varepsilon\|_{p,I_T} \leq c\|\kappa^\varepsilon\|_{p,I_T} \leq cT^{1/p},$$

hence the result follows. □

7 Proof of the main theorem

At this stage, we are ready to present the proof of our main result (Theorem 1.1). This depends essentially on the passage to the limit in the family of solutions $(\rho^\varepsilon, \kappa^\varepsilon)$ of system (2.2). Since $\kappa^\varepsilon_x \neq 0$, we multiply the first equation of (2.2) by κ^ε_x and we rewrite system (2.2) in terms of A^ε, we obtain:

$$\begin{cases} \kappa^\varepsilon_t \kappa^\varepsilon_x = \varepsilon\kappa^\varepsilon_x\kappa^\varepsilon_{xx} + \rho^\varepsilon_x A^\varepsilon_x & \text{on} \quad I_T \\ \rho^\varepsilon_t = \varepsilon\rho^\varepsilon_{xx} + A^\varepsilon_x & \text{on} \quad I_T. \end{cases} \tag{7.1}$$

We will pass to the limit in the framework of viscosity solutions for the first equation of (7.1), and in the distributional sense for the second equation. We start with the following proposition.

Proposition 7.1 ***(Local uniform convergence)***
The sequences $(\rho^\varepsilon)_\varepsilon$, $(\rho^\varepsilon_x)_\varepsilon$, $(\kappa^\varepsilon)_\varepsilon$, $(A^\varepsilon)_\varepsilon$ and $(A^\varepsilon_x)_\varepsilon$ converge (up to extraction of a subsequence) locally uniformly in I_T as ε goes to zero.

Proof. Let V be an open compactly contained subset of I_T. The constants that will appear in the proof are all independent of ε. However, they may depend on other fixed parameters including V. The idea is to give an ε-uniform control of the modulus of continuity in space and in time of the quantities mentioned in Proposition (7.1), which gives the local uniform convergence. The ε-uniform control on the space modulus of continuity will be derived from the Corollary 5.2 and Lemma 5.4, while the ε-uniform control on the time modulus of continuity will be derived from Lemma 3.4. The proof is divided into five steps.

Step 1. (Convergence of A^ε and A^ε_x)

From (3.5), we know that $\left\|\frac{\rho^\varepsilon_x}{\kappa^\varepsilon_x}\right\|_\infty \le 1$. We apply the interior L^p, $p > 1$, estimates for parabolic equations (see for instance [19, Theorem 7.13, page 172]) to the term A^ε satisfying (6.2), we obtain:

$$\|A^\varepsilon\|_{W^{2,1}_p(V)} \le c_3\|A^\varepsilon\|_{p,V'}, \tag{7.2}$$

where V' is any open subset of I_T satisfying $V \subset\subset V' \subset\subset I_T$. The constant $c_3 = c_3(p,\tau,V,V')$ can be chosen independent of ε first by applying a parabolic rescaling of (6.2), and then using the fact that the factor multiplied by A^ε_{xx} in (6.2) satisfying $1 \le 1+\varepsilon \le 2$. At this point, we apply Lemma 6.2 for A^ε on V', we get:

$$\|A^\varepsilon\|_{p,V'} \le c_4(\|A^\varepsilon\|_{1,I_T} + 1), \tag{7.3}$$

and hence the above two equations (7.2) and (7.3) give:

$$\|A^\varepsilon\|_{W^{2,1}_p(V)} \le c_5(\|A^\varepsilon\|_{1,I_T} + 1). \tag{7.4}$$

We estimate the right hand side of (7.4) in the following way:

$$\begin{aligned}
\|A^\varepsilon\|_{1,I_T} &= \int_{I_T} |\rho^\varepsilon_x - \tau\kappa^\varepsilon| \\
&\le \int_{I_T} \kappa^\varepsilon_x + \tau|\kappa^\varepsilon| \\
&\le (1+\tau)T,
\end{aligned}$$

where we have used the fact that $|\rho^\varepsilon_x| < \kappa^\varepsilon_x$ (see (3.5) of Theorem 3.1) in the second line, and the fact that $0 \le \kappa^\varepsilon \le 1$ (see Remark 3.3) in the last line. Therefore, inequality (7.4) implies:

$$\|A^\varepsilon\|_{W^{2,1}_p(V)} \le c_6, \quad 1 < p < \infty. \tag{7.5}$$

We use the above inequality for $p > 3$. In this case, the Sobolev embedding in Hölder spaces (see [16, Lemma 2.8]) gives:

$$W^{2,1}_p(V) \hookrightarrow C^{1+\alpha,\frac{1+\alpha}{2}}(V), \quad \alpha = 1 - 3/p$$

and hence (7.5) implies:

$$\|A^\varepsilon\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(V)} \le c_7, \tag{7.6}$$

which guarantees the equicontinuity and the equiboundedness of $(A^\varepsilon)_\varepsilon$ and $(A^\varepsilon_x)_\varepsilon$. By the Arzela-Ascoli Theorem (see for instance [3]), we finally obtain

$$A^\varepsilon \longrightarrow A \quad \text{and} \quad A^\varepsilon_x \longrightarrow A_x, \tag{7.7}$$

up to a subsequence, uniformly on V as $\varepsilon \to 0$.

Step 2. (Convergence of κ^ε)

We control the modulus of continuity of κ^ε in space and in time, locally uniformly with respect to ε.

Step 2.1. (Control of the modulus of continuity in time)

The first equation of (7.1) gives:

$$\kappa_t^\varepsilon = \varepsilon\kappa_{xx}^\varepsilon + \frac{\rho_x^\varepsilon}{\kappa_x^\varepsilon}A_x^\varepsilon,$$

and hence, using the fact that $\left\|\frac{\rho_x^\varepsilon}{\kappa_x^\varepsilon}\right\|_\infty \leq 1$, together with (7.6), we get:

$$\|\kappa_t^\varepsilon - \varepsilon\kappa_{xx}^\varepsilon\|_{\infty,V} \leq \left\|\frac{\rho_x^\varepsilon}{\kappa_x^\varepsilon}\right\|_{\infty,V}\|A_x\|_{\infty,V} \leq c_7. \tag{7.8}$$

Also, by (3.6), we have:

$$\|\kappa^\varepsilon\|_{\infty,V} \leq 1.$$

This uniform bound on κ^ε together with (7.8) permit to use Lemma 3.4 to conclude that

$$|\kappa^\varepsilon(x,t) - \kappa^\varepsilon(x,t+h)| \leq c_8 h^\beta, \quad (x,t),(x,t+h)\in V, \quad 0<\beta<1, \tag{7.9}$$

which controls the modulus of continuity of κ^ε with respect to t uniformly in ε. We now move to control the moduls of continuity in space.

Step 2.2 (An ε-uniform bound on $S(0)$)

Recall the definition (5.2) of $S(t)$:

$$S(t) = \int_I \sum_\pm \theta^{\pm,\varepsilon}(x,t)\log\theta^{\pm,\varepsilon}(x,t)dx,$$

with

$$\theta^{\pm,\varepsilon} = \frac{\kappa_x^\varepsilon \pm \rho_x^\varepsilon}{2}.$$

Hence

$$S(0) = \int_I \frac{\kappa_x^{0,\varepsilon}+\rho_x^{0,\varepsilon}}{2}\log\left(\frac{\kappa_x^{0,\varepsilon}+\rho_x^{0,\varepsilon}}{2}\right) + \int_I \frac{\kappa_x^{0,\varepsilon}-\rho_x^{0,\varepsilon}}{2}\log\left(\frac{\kappa_x^{0,\varepsilon}-\rho_x^{0,\varepsilon}}{2}\right).$$

Using the elementarty identity $x\log x \leq x^2$ and $(x\pm y)^2 \leq 2(x^2+y^2)$, we compute:

$$\begin{aligned} S(0) &\leq \int_I \left(\frac{\kappa_x^{0,\varepsilon}+\rho_x^{0,\varepsilon}}{2}\right)^2 + \int_I \left(\frac{\kappa_x^{0,\varepsilon}-\rho_x^{0,\varepsilon}}{2}\right)^2 \\ &\leq \|\rho_x^{0,\varepsilon}\|_{2,I}^2 + \|\kappa_x^{0,\varepsilon}\|_{2,I}^2. \end{aligned} \tag{7.10}$$

From (4.1) and (4.2), we know that:

$$|\rho_x^{0,\varepsilon}| = \left| \frac{\rho_x^0 + \varepsilon\tau\phi'}{(1+\varepsilon)^2} \right| \leq \frac{|\rho_x^0| + \varepsilon}{(1+\varepsilon)^2} \leq |\rho_x^0| + 1,$$

and

$$|\kappa_x^{0,\varepsilon}| = \left| \frac{\kappa_x^0 + \varepsilon}{1+\varepsilon} \right| \leq |\kappa_x^0| + 1.$$

Using the above two inequalities into (7.10), we deduce that:

$$S(0) \leq 2(\|\rho_x^0\|_{2,I}^2 + \|\kappa_x^0\|_{2,I}^2 + 2).$$

Step 2.3. (Control of the modulus of continuity in space and conclusion)

We use the uniform bound obtained for $S(0)$ in Step 2.1, together with the special control (5.6) of κ_x^ε given in Corollary 5.2, we get for all $0 \leq t \leq T$:

$$\int_I \kappa_x^\varepsilon(x,t) \log(\kappa_x^\varepsilon(x,t))dx \leq 2(\|\rho_x^0\|_{2,I}^2 + \|\kappa_x^0\|_{2,I}^2 + 2) + \frac{\tau^2 T}{4} + 1,$$

therefore

$$\int_I \kappa_x^\varepsilon(x,t) \log(\kappa_x^\varepsilon(x,t))dx \leq c_9, \quad \forall\, 0 \leq t \leq T. \tag{7.11}$$

Inequality (7.11) permit to use Lemma 5.4, hence we obtain:

$$|\kappa^\varepsilon(x+h,t) - \kappa^\varepsilon(x,t)| \leq \frac{c_{10}}{|\log h|}, \quad (x,t),(x+h,t) \in I_T, \tag{7.12}$$

Inequalities (7.9) and (7.12) give the equicontinuity of the sequence $(\kappa^\varepsilon)_\varepsilon$ on V, and again by the Arzela-Ascoli Theorem, we get:

$$\kappa^\varepsilon \to \kappa, \tag{7.13}$$

up to a subsequence, uniformly on V as $\varepsilon \to 0$.

Step 3. (Convergence of ρ^ε)

As in step 2, we control the modulus of continuity of ρ^ε in space and in time, locally uniformly with respect to ε.

Step 3.1. (Control of the modulus of continuity in time)

The second equation of (7.1) gives:

$$\rho_t^\varepsilon - \varepsilon\rho_{xx}^\varepsilon = A_x^\varepsilon,$$

hence, from (7.6), we deduce that:

$$\|\rho_t^\varepsilon - \varepsilon\rho_{xx}^\varepsilon\|_{\infty,V} \leq c_7,$$

and from (3.6), we have:

$$\|\rho^\varepsilon\|_{\infty,V} \leq 1.$$

The above two inequalities permit to use Lemma 3.4, we finally get:

$$|\rho^\varepsilon(x,t) - \rho^\varepsilon(x,t+h)| \leq c_8 h^\beta, \quad (x,t),(x,t+h) \in V, \quad 0 < \beta < 1, \tag{7.14}$$

which controls the modulus of continuity of ρ^ε with respect to t uniformly in ε.

Step 3.2. (Control of the modulus of continuity in space and conclusion)

The control of the space modulus of continuity is based on the following observation. From (3.5), we know that $|\rho^\varepsilon_x| \leq \kappa^\varepsilon_x$ on I_T. Using this inequality, we get, for every $(x,t),(x+h,t) \in I_T$:

$$|\rho^\varepsilon(x+h,t) - \rho^\varepsilon(x,t)| \leq \int_x^{x+h} |\rho^\varepsilon_x(y,t)|dy \leq \int_x^{x+h} \kappa^\varepsilon_x(y,t)dy \leq |\kappa^\varepsilon(x+h,t) - \kappa^\varepsilon(x,t)|.$$

Inequality (7.12) gives immediately that:

$$|\rho^\varepsilon(x+h,t) - \rho^\varepsilon(x,t)| \leq \frac{c_{10}}{|\log h|}, \quad (x,t),(x+h,t) \in I_T. \tag{7.15}$$

From (7.14) and (7.15), we deduce that:

$$\rho^\varepsilon \to \rho, \tag{7.16}$$

up to a subsequence, uniformly on V as $\varepsilon \to 0$.

Step 4. (Convergence of ρ^ε_x and conclusion)

In fact, this follows from Step 1, Step 2, and the fact that

$$\rho^\varepsilon_x = A^\varepsilon + \tau\kappa^\varepsilon \to \rho_x, \tag{7.17}$$

uniformly on V as $\varepsilon \to 0$. In this case, we also deduce that

$$A = \rho_x - \tau\kappa.$$

The proof of Proposition 7.1 is done. □

We now move to the proof of the main result.

Proof of Theorem 1.1. We first remark that κ^ε is a viscosity solution of the first equation of (7.1):

$$\kappa^\varepsilon_t \kappa^\varepsilon_x - \varepsilon\kappa^\varepsilon_x\kappa^\varepsilon_{xx} - \rho^\varepsilon_x A^\varepsilon_x = 0 \quad \text{on} \quad I_T. \tag{7.18}$$

Indeed, let $\phi \in C^2(I_T)$ such that $\kappa^\varepsilon - \phi$ has a local maximum at some point $(x_0,t_0) \in I_T$. Then $D\kappa^\varepsilon = D\phi$ and $D^2\kappa^\varepsilon \leq D^2\phi$. From this and the fact that $\kappa^\varepsilon_x > 0$, we calculate at (x_0,t_0):

$$\begin{aligned} \phi_t\phi_x - \varepsilon\phi_x\phi_{xx} - \rho^\varepsilon_x A^\varepsilon_x &= \kappa^\varepsilon_t\kappa^\varepsilon_x - \varepsilon\kappa^\varepsilon_x\phi_{xx} - \rho^\varepsilon_x A^\varepsilon_x \\ &\leq \kappa^\varepsilon_t\kappa^\varepsilon_x - \varepsilon\kappa^\varepsilon_x\kappa^\varepsilon_{xx} - \rho^\varepsilon_x A^\varepsilon_x \\ &\leq 0. \end{aligned}$$

On the other hand, if $\kappa^\varepsilon - \phi$ has a local minimum at (x_0, t_0), we similarly get:

$$\phi_t \phi_x - \varepsilon \phi_x \phi_{xx} - \rho_x^\varepsilon A_x^\varepsilon \geq 0,$$

and hence κ^ε is a viscosity solution.

Remark 7.2 *The equation (7.18) can be viewed as the following Hamilton-Jacobi equation of second order:*

$$H^\varepsilon(X, D\kappa^\varepsilon, D^2\kappa^\varepsilon) = 0, \quad X = (x,t) \in I_T \tag{7.19}$$

with

$$D\kappa^\varepsilon = (\kappa_x^\varepsilon, \kappa_t^\varepsilon) \quad \text{and} \quad D^2\kappa^\varepsilon = \begin{pmatrix} \kappa_{xx}^\varepsilon & \kappa_{xt}^\varepsilon \\ \kappa_{tx}^\varepsilon & \kappa_{tt}^\varepsilon \end{pmatrix},$$

where H^ε is the Hamiltonian function given by:

$$\begin{array}{rcl} H^\varepsilon : I_T \times \mathbb{R}^2 \times M_{sym}^{2\times 2} & \longrightarrow & \mathbb{R} \\ (X, p, M) & \longmapsto & H^\varepsilon(X,p,M) = p_1 p_2 - \varepsilon p_1 M_{11} - \rho_x^\varepsilon(X) A_x^\varepsilon(X), \end{array} \tag{7.20}$$

$p = (p_1, p_2)$ and $M = (M_{ij})_{i,j=1,2}$.

From (7.7) and (7.17), we deduce that $(H^\varepsilon)_\varepsilon$ converges locally uniformly in $I_T \times \mathbb{R}^2 \times M_{sym}^{2\times 2}$ to the function H given by:

$$\begin{array}{rcl} H : I_T \times \mathbb{R}^2 \times M_{\text{sym}}^{2\times 2} & \longrightarrow & \mathbb{R} \\ (X, p, M) & \longmapsto & H(X,p,M) = p_1 p_2 - \rho_x(X) A_x(X). \end{array} \tag{7.21}$$

This, together with the local uniform convergence of κ^ε to κ (see 7.13), and the fact that κ^ε is a viscosity solution of (7.18), permit to use the stability of viscosity solutions (see Theorem 3.7), which proves that κ is a viscosity solution of

$$H(X, D\kappa, D^2\kappa) = \kappa_t \kappa_x - \rho_x A_x = 0 \quad \text{in} \quad I_T. \tag{7.22}$$

We now pass to the limit $\varepsilon \to 0$ in the second equation of (7.1), we obtain

$$\rho_t = A_x \quad \text{in} \quad \mathcal{D}'(I_T). \tag{7.23}$$

From (7.22) and (7.23), we get:

1. κ is a viscosity solution of $\kappa_t \kappa_x = \rho_t \rho_x$ in I_T;

2. ρ is a distributional solution of $\rho_t = \rho_{xx} - \tau \kappa_x$ in I_T.

Let us now prove inequality (1.13). Let $\phi \in C_0^\infty(I_T)$ be a non-negative test function. From (3.5), we know that

$$\kappa_x^\varepsilon > |\rho_x^\varepsilon| \quad \text{in} \quad I_T,$$

and hence

$$\kappa_x^\varepsilon > \rho_x^\varepsilon \quad \text{and} \quad \kappa_x^\varepsilon > -\rho_x^\varepsilon \quad \text{in} \quad I_T.$$

Multiplying these inequalities by a test function $\phi \in \mathcal{D}(I_T)$, $\phi \geq 0$; integrating by parts over I_T, and passing to the limit as $\varepsilon \to 0$, we obtain

$$\kappa_x \geq \rho_x \quad \text{and} \quad \kappa_x \geq -\rho_x \quad \text{in} \quad \mathcal{D}'(I_T),$$

therefore

$$\kappa_x \geq |\rho_x| \quad \text{in} \quad \mathcal{D}'(I_T).$$

Finally, let us show that the two solutions ρ and κ can be extended by continuity to the parabolic boundary of I_T, in order to retrieve the initial and boundary conditions. Indeed, the local uniform convergence $(\rho^\varepsilon, \kappa^\varepsilon) \to (\rho, \kappa)$, together with the uniform control of the modulus of continuity of these solutions:

- with respect to x near $\partial I \times [0,T]$ by (7.12);
- with respect to t near $I \times \{t = 0\}$, away from 0 and 1 by (7.9),

and the fact that $\kappa^{0,\varepsilon} \to \kappa^0$, $\rho^{0,\varepsilon} \to \rho^0$ uniformly in $\bar{I}$,

$$\kappa^\varepsilon(0,.) \to 0, \quad \kappa^\varepsilon(1,.) = 1, \quad \rho^\varepsilon = 0 \quad \text{on} \quad \partial I \times [0,T],$$

show that $(\rho, \kappa) \in (C(\overline{I_T}))^2$, so the initial and boundary conditions are satisfied pointwisely, and the proof of the main result is done. $\square$

8 Application: simulations for the evolution of elastoviscoplatic materials

Motivated by the simulation of the elastoviscoplastic behavior that are formulated by the model of Groma, Csikor and Zaiser [12], this section is devoted to write down the equations of the displacement vector u inside the crystal when it is applied to a constant exterior shear stress τ on the boundary walls (see Figure 1). Also, at the end of this section, we present some numerical simulations revealing the evolution of a crystal of small size.

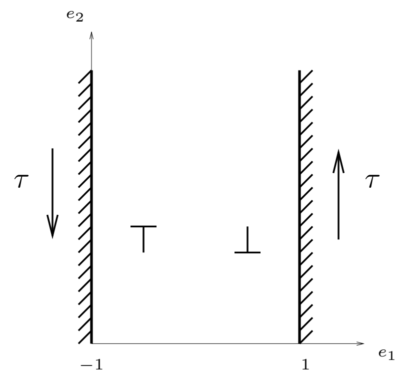

Figure 1: Geometry of the crystal.

Here, as we have already mentioned in the introduction, we suppose that the distribution of dislocations is invariant by translation in the y-direction. Also, we assume, without loss of generality (up to a change of variables in (x,t) and a re-definition of τ), that

$$I = (-1, 1).$$

We consider a 2-dimentional crystal (Figure 1) with the displacement vector:

$$u = (u_1, u_2) : \mathbb{R}^2 \longmapsto \mathbb{R}^2.$$

For $x = (x_1, x_2)$ and an orthonormal basis (e_1, e_2), we define the total strain by:

$$\varepsilon(u) = \frac{1}{2}(\nabla u + {}^t\nabla u), \tag{8.1}$$

i.e.

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\partial_j u_i + \partial_i u_j\right)$$

with

$$\partial_j u_i = \frac{\partial u_i}{\partial x_j}, \quad i, j = 1, 2.$$

This total strain can be decomposed into two parts as follows:

$$\varepsilon(u) = \varepsilon^e(u) + \varepsilon^p, \tag{8.2}$$

where $\varepsilon^e(u)$ is the elastic strain and ε^p is the plastic strain which is given by:

$$\varepsilon^p = \gamma\varepsilon^0, \tag{8.3}$$

with

$$\varepsilon^0 = \frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

in the special case of a single slip system where dislocations move following the Burgers vector $\vec{b} = e_1$. Here γ is the resolved plastic strain that can be expressed in terms of the dislocation densities as:

$$\gamma = \rho^+ - \rho^- = \rho,$$

therefore (8.3) implies that

$$\varepsilon^p = \rho\varepsilon^0.$$

The stress field σ inside the crystal is given by:

$$\sigma = \Lambda : \varepsilon^e(u),$$

where for $i, j = 1, 2$,

$$\sigma_{ij} = (\Lambda : \varepsilon^e(u))_{ij} = 2\mu\varepsilon^e_{ij}(u) + \lambda\delta_{ij}tr(\varepsilon^e(u)), \tag{8.4}$$

with $\lambda, \mu > 0$ are the constants of Lamé coefficients of the crystal that are assumed (for simplification) to be isotropic, and δ_{ij} is the Kronecker delta symbol. This stress field σ has to satisfy the equation of elasticity:

$$\mathrm{div}\sigma = 0. \tag{8.5}$$

Finally, the functions ρ, κ (solutions of (1.3)) and u are solutions of the following coupled system:

$$\begin{cases} \mathrm{div}\sigma &= 0 & \text{in} \quad I\times(0,\infty), \\ \sigma &= \Lambda:(\varepsilon(u)-\varepsilon^p) & \text{in} \quad I\times(0,\infty), \\ \varepsilon(u) &= \dfrac{1}{2}(\nabla u + {}^t\nabla u) & \text{in} \quad I\times(0,\infty), \\ \varepsilon^p &= \varepsilon^0(\rho^+-\rho^-) & \text{in} \quad I\times(0,\infty), \\ \kappa_t\kappa_x &= \rho_t\rho_x & \text{in} \quad I\times(0,\infty), \\ \rho_t &= \rho_{xx}-\tau\kappa_x & \text{in} \quad I\times(0,\infty), \end{cases} \tag{8.6}$$

Equation (8.5) can be reformulated as:

$$\mathrm{div}\,(2\mu\varepsilon(u)+\lambda tr(\varepsilon(u))I_d) = \mathrm{div}\,(2\mu\varepsilon^p+\lambda tr(\varepsilon^p)I_d),$$

which implies that:

$$\mu\Delta u + (\lambda+\mu)\nabla(\mathrm{div}\, u) = \mu\begin{pmatrix} \partial_2\rho \\ \partial_1\rho \end{pmatrix} = \mu\begin{pmatrix} 0 \\ \partial_1\rho \end{pmatrix}. \tag{8.7}$$

Here $\partial_2\rho = 0$ is due to the homogeneity of the distribution of dislocations in the e_2-direction.

Calculation of u. We first calculate the value of the displacement u on the boundary walls. Remark first that since we are applying a constant shear stress field on the walls, the stress field σ there can be evaluated as: $\sigma\cdot n = \pm\tau e_2$, $n = \pm e_1$,

$$\sigma^{\mathrm{b}} = \begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix}, \quad \text{on} \quad \partial I. \tag{8.8}$$

Using (8.8) and (8.4), we can derive the following equations on the boundary:

$$\begin{cases} \partial_1 u_1 = 0 & \text{on} \quad \partial I, \\ \mu(\partial_1 u_2 - \rho) = \tau & \text{on} \quad \partial I. \end{cases} \tag{8.9}$$

Equation (8.7) leads to the following two equations inside I:

$$\begin{cases} \partial_1[(\lambda+2\mu)\partial_1 u_1] = 0 & \text{on} \quad I \\ \partial_1(\partial_1 u_2 - \rho) = 0 & \text{on} \quad I. \end{cases} \tag{8.10}$$

Combining (8.9) and (8.10) we deduce that:

$$\begin{cases} \partial_1 u_1 = 0 & \text{on} \quad I \\ \partial_1 u_2 - \rho = \dfrac{\tau}{\mu} & \text{on} \quad I. \end{cases} \tag{8.11}$$

By the antisymmetry of our particular configuration with respect to the line $x_1 = 0$, and the fact that we are applying a shear stress on the walls, we eventually have:

$$u_1(0,x_2) = u_2(0,x_2) = 0,$$

which together with (8.11) finally lead:

$$\begin{cases} u_1(x_1,x_2) = 0, & (x_1,x_2) \in I \times \mathbb{R} \\ u_2(x_1,x_2) = \dfrac{\tau}{\mu}x_1 + \displaystyle\int_0^{x_1} \rho(x)dx, & (x_1,x_2) \in I \times \mathbb{R}. \end{cases} \tag{8.12}$$

As an elastoviscoplastic material of small size, the double-ended pile-up distribution of dislocations affects the internal contribution (displacement) of the material near the boundary (see Figures 2, 3 and 4). It appears that the crystal is perfectly elastic at a very small time $t = 0^+$, while the plastic contribution starts to take place at $t > 0$ with two boundary layers created at the walls (see Figure 4). The following figures are numerically computed after calculating the displacement u_2 (see (8.12)) by discretizing the last two equations of (8.6) in order to calculate ρ.

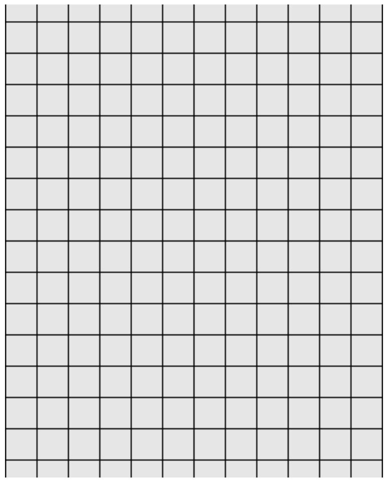

Figure 2: The material at $t = 0$.

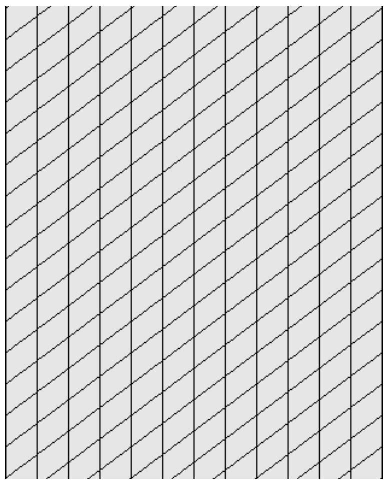

Figure 3: The elastic deformation at $t = 0^+$.

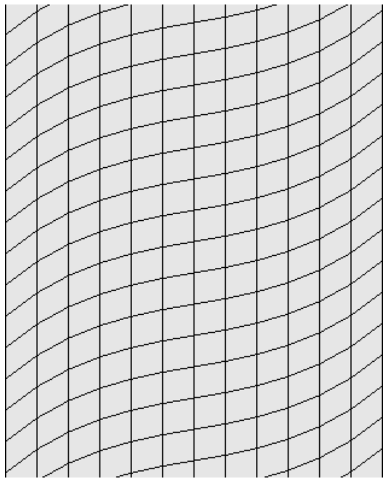

Figure 4: The total deformation at $t = +\infty$.

9 Appendix

A1. Proof of Lemma 3.4 (control of the modulus of continuity in time)

Let V be a compactly contained subset of I_T. Throughout the proof, the constant c may take several values but only depending on V. Since $V \subset\subset I_T$, then there is a rectangular cube of the form

$$\mathcal{Q} = (x_1, x_2) \times (t_1, t_2),$$

such that $V \subset\subset \mathcal{Q} \subset\subset I_T$. In this case, there exists a constant ε_0, also depending on V such that for any

$$0 < \varepsilon < \varepsilon_0,$$

and any $(x,t) \in V$, we have:

$$(x - 2\sqrt{\varepsilon}, x + 2\sqrt{\varepsilon}) \times \{t\} \subset \mathcal{Q}.$$

Moreover, for any $(x,t), (x, t+h) \in V$, we can always find two intervals $\mathcal{I}$ and $\mathcal{J}$ such that

$$(t, t+h) \subset \mathcal{I} \subset\subset \mathcal{J},$$

with

$$\{x\} \times \mathcal{I} \subset \mathcal{Q} \quad \text{and} \quad \{x\} \times \mathcal{J} \subset \mathcal{Q}.$$

Let us indicate that these intervals might have different lengths depending on h and V but we always have

$$|\mathcal{J}|, |\mathcal{I}| \leq |t_2 - t_1|.$$

Consider the following rescaling of the function u^ε defined by:

$$\tilde{u}^\varepsilon(x,t) = u^\varepsilon(\sqrt{\varepsilon}x, t). \tag{9.1}$$

This function satisfies

$$\tilde{u}^\varepsilon_t = \tilde{u}^\varepsilon_{xx} + \tilde{f}^\varepsilon, \quad (x,t) \in (0, 1/\sqrt{\varepsilon}) \times (0,T),$$

where $\tilde{f}^\varepsilon(x,t) = f^\varepsilon(\sqrt{\varepsilon}x, t)$. Take (x_0,t_0), (x_0, t_0+h) in V, and let

$$\mathcal{Q}_1 = (x_0 - \sqrt{\varepsilon}, x_0 + \sqrt{\varepsilon}) \times \mathcal{I} \quad \text{and} \quad \mathcal{Q}_2 = (x_0 - 2\sqrt{\varepsilon}, x_0 + 2\sqrt{\varepsilon}) \times \mathcal{J}.$$

These two cylinders are transformed by the above rescaling into

$$\tilde{\mathcal{Q}}_1 = \left(\frac{x_0}{\sqrt{\varepsilon}} - 1, \frac{x_0}{\sqrt{\varepsilon}} + 1\right) \times \mathcal{I} \quad \text{and} \quad \tilde{\mathcal{Q}}_2 = \left(\frac{x_0}{\sqrt{\varepsilon}} - 2, \frac{x_0}{\sqrt{\varepsilon}} + 2\right) \times \mathcal{J}.$$

We apply the interior L^p, $p > 3$, estimates for parabolic equations (see for instance [19, Theorem 7.13, page 172]) to the function $\tilde{u}^\varepsilon$ over the domains $\tilde{\mathcal{Q}}_1 \subset\subset \tilde{\mathcal{Q}}_2$, we get

$$\|\tilde{u}^\varepsilon\|_{W^{2,1}_p(\tilde{\mathcal{Q}}_1)} \le c(\|\tilde{u}^\varepsilon\|_{p,\tilde{\mathcal{Q}}_2} + \|\tilde{f}^\varepsilon\|_{p,\tilde{\mathcal{Q}}_2}). \tag{9.2}$$

We compute:

$$\begin{aligned}
\|\tilde{u}^\varepsilon\|^p_{L^p(\tilde{\mathcal{Q}}_2)} &= \int_{\tilde{\mathcal{Q}}_2} |\tilde{u}^\varepsilon(x,t)|^p dxdt \\
&= \int_{\tilde{\mathcal{Q}}_2} |u^\varepsilon(\sqrt{\varepsilon}x,t)|^p dxdt \\
&= \frac{1}{\sqrt{\varepsilon}} \int_{\mathcal{Q}_2} |u^\varepsilon(y,t)|^p dydt \\
&\le c,
\end{aligned} \tag{9.3}$$

where for the last line, we have used the local uniform boundedness of $(u^\varepsilon)_\varepsilon$, and in exactly the same way (from the local uniform boundedness of and $(f^\varepsilon)_\varepsilon$) we obtain:

$$\|\tilde{f}^\varepsilon\|^p_{L^p(\tilde{\mathcal{Q}}_2)} \le c. \tag{9.4}$$

Therefore, from (9.3), (9.4), inequality (9.2) implies:

$$\|\tilde{u}^\varepsilon\|_{W^{2,1}_p(\tilde{\mathcal{Q}}_1)} \le c. \tag{9.5}$$

We use the Sobolev embedding in Hölder spaces (see for instance [16, Lemma 2.8]):

$$W^{2,1}_p(\tilde{\mathcal{Q}}_1) \hookrightarrow C^{1+\alpha\,\frac{1+\alpha}{2}}(\tilde{\mathcal{Q}}_1), \quad p > 3,\ \alpha = 1 - 3/p,$$

to obtain, from (9.5), that:

$$\|\tilde{u}^\varepsilon\|_{C^{1+\alpha\,\frac{1+\alpha}{2}}(\tilde{\mathcal{Q}}_1)} \le c,$$

and hence

$$\frac{|\tilde{u}^\varepsilon(x_0/\sqrt{\varepsilon}, t_0+h) - \tilde{u}^\varepsilon(x_0/\sqrt{\varepsilon}, t_0)|}{h^{\frac{1+\alpha}{2}}} \le c,$$

then from (9.1),

$$\frac{|u^\varepsilon(x_0, t_0+h) - u^\varepsilon(x_0,t_0)|}{h^{\frac{1+\alpha}{2}}} \le c.$$

Choosing $\beta = \frac{1+\alpha}{2}$ we get the desired result. $\square$

A2. Proof of Lemma 3.5 (An interior estimate for the heat equation)

Recall that a is a solution of the heat equation on I_T,

$$a_t = a_{xx}.$$

The proof of lemma 3.5 depends mainly on a mean value formula for solutions of the heat equations. Usually, basic mean value formulae of the solution of the heat equation are expressed through unbounded kernels (see for example [7, Theorem 1]), where a can be expressed as:

$$a(x_0, t_0) = (4\pi r^2)^{-1/2} \int_{\Omega_r(x_0,t_0)} a(x,t) \frac{(x_0 - x)^2}{4(t_0 - t)^2} dxdt. \tag{9.6}$$

Here, $(x_0, t_0) \in I_T$, $(x, t) \in \Omega_r(X_0)$, and $r > 0$ small enough in order to ensure that the parabolic ball of radius r:

$$\Omega_r(x_0, t_0) = \left\{ (x,t);\ t_0 - r^2 < t < t_0,\ \ (x - x_0)^2 < 2(t_0 - t) \log\left(\frac{r^2}{t_0 - t}\right) \right\} \subset I_T. \tag{9.7}$$

In our case, we need a mean value formula similar to (9.6) but with a bounded kernel on $\Omega_r(x_0, t_0)$. In [10], the authors have given such a representation formula for the solution of the heat equation. We present their result in a simplified version.

Theorem 9.1 ***(Mean value formula with bounded kernels, [10, Theorem 3.1])*** *Let $u \in C^2(\mathcal{D})$ be a solution of the heat equation:*

$$u_t = u_{xx} \quad on \quad \mathcal{D},$$

where $\mathcal{D}$ is an oben subset of $\mathbb{R}^2$ containing the modified unit parabolic ball $\Omega'_1(0,0)$, with

$$\Omega'_1(0,0) = \{(x,t);\ -1 < t < 0,\ \ x^2 < 8t \log(-t)\}.$$

Then we have:

$$u(0,0) = \int_{\Omega'_1(0,0)} u(x,t) E(x,t) dxdt, \tag{9.8}$$

where the kernel E satisfies:

$$\|E(x,t)\|_{\infty, \Omega'_1(0,0)} \leq c, \tag{9.9}$$

and $c > 0$ is a fixed positive constant.

Remark 9.2 *The above Theorem is an application of [10, Theorem 3.1] in the case $m = 3$. In this case, an explicit expression of E is given by:*

$$E(x,t) = \frac{\omega_3}{16\pi^2} \left(-x^2 + 8t \log(-t)\right)^{3/2} \left[\frac{x^2}{4t^2} + \frac{3(-x^2 + 8t \log(-t))}{20t^2} \right],$$

where ω_3 is the volume of the unit ball in $\mathbb{R}^3$. For a more general expression of E, we send the reader to [10, Equality (3.6) of Theorem 3.1].

Using the parabolic rescaling, we can obtain a similar mean value representation at any $(x_0, t_0) \in \mathbb{R}^2$. More precisely, we have:

Corollary 9.3 ***(Mean value formula at any point*** $(x_0, t_0) \in \mathbb{R}^2$***)***
Let $u \in C^2(\mathcal{D})$ be a solution of the heat equation:

$$u_t = u_{xx} \quad on \quad \mathcal{D},$$

where $\mathcal{D}$ is an oben subset of $\mathbb{R}^2$ containing the modified unit parabolic ball $\Omega'_r(x_0, t_0)$, $r > 0$, with

$$\Omega'_r(x_0, t_0) = \left\{ (x, t);\ t_0 - r^2 < t < t_0,\ \ |x - x_0|^2 < 8(t_0 - t) \log\left(\frac{r^2}{t_0 - t}\right) \right\}.$$

Then we have:

$$u(x_0, t_0) = \frac{\bar{c}}{|\Omega'_r(x_0, t_0)|} \int_{\Omega'_r(x_0,t_0)} u(x, t) E\left(\frac{x - x_0}{r}, \frac{t - t_0}{r^2}\right) dxdt, \tag{9.10}$$

where $\bar{c} > 0$ and $|\Omega'_r(x_0, t_0)| = \bar{c}r^3$.

Back to the proof of Lemma 3.5. Since $V \subset\subset I_T$, then there exists a fixed

$$r_0 = r_0(\mathrm{dist}(V, \partial^p I_T)),$$

such that:

$$\Omega'_{r_0}(x_0, t_0) \subset I_T, \quad \forall\ (x_0, t_0) \in V.$$

We use the mean value formula (9.10) at the point (x_0, t_0), we obtain:

$$a(x_0, t_0) = r_0^{-3} \int_{\Omega'_{r_0}(x_0,t_0)} a(x, t) E\left(\frac{x - x_0}{r_0}, \frac{t - t_0}{r_0^2}\right) dxdt,$$

and hence from the L^∞ bound (9.9)of E on $\Omega'_1(0, 0)$, we deduce that:

$$\|a\|_{\infty,V} \leq c r_0^{-3} \|a\|_{1,I_T},$$

where the constant c is given by (9.9). Finally, we obtain:

$$\|a\|_{p,V} \leq c r_0^{-3} |V|^{1/p} \|a\|_{1,I_T},$$

and the result follows. □

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